

Last time Finite subgrps of O_2

Recall: M_2 denotes the gp of isometries of $\mathbb{R}^2$

Will: classify the finite subgrps of M_2

Lemma: Suppr $P_1, \dots, P_n \in \mathbb{R}^2$, $P = \frac{P_1 + P_2 + \dots + P_n}{n}$.

If $f \in M_2$, then

$$f(P) = \frac{f(P_1) + \dots + f(P_n)}{n}$$

Pf: Any elt $g \in M_2$ is of form $g = t_a \circ \rho$ for some $a \in \mathbb{R}^2$, $\rho \in \mathbb{R}$. To prove the lemma, suffices to prove it for ρ and t_a individually.

• Clear for ρ because ρ is linear

• For t_a , have

$$\frac{t_a(P_1) + \dots + t_a(P_n)}{n} = \frac{(P_1 + a) + \dots + (P_n + a)}{n} = \frac{P_1 + \dots + P_n}{n} + a$$

$$= P + a = t_a(P)$$

□

Thm Suppr $G \subseteq M_2$ is finite subgp. Then $\exists P \in \mathbb{R}^2$ s.t.
 $g(P) = P$ for every $g \in G$.

Pf: Take arbitrary pt $Q \in \mathbb{R}^2$. Define

Pf: Take arbitrary pt $Q \in \mathbb{R}^d$. Define

$$P = \frac{1}{|G|} \sum_{g \in G} g(Q).$$

Then, if $h \in G$, then $h(P) = \frac{1}{|G|} \sum_{g \in G} h g(Q)$

$$= \frac{1}{|G|} \sum_{g' \in G} g'(Q) = P.$$

□

Cor: Suppose $G \subseteq M_2$ is finite. Then $G \cong C_n$ or D_n for some n .

Pf: By Thm, $\exists P \in \mathbb{R}^2$ so that $g(P) = P$ for all $g \in G$.

Consider $G' = t_{-P} G t_{+P} \subseteq M_2$.

Then if $g' \in G'$, $g'(0) = 0$: if $g' \in G'$, then

$g' = t_{-P} g t_P$ for some $g \in G$. Thus

$$g'(0) = t_{-P} g(P) = t_{-P}(P) = 0.$$

$\Rightarrow G' \subseteq O_2$ is finite $\Rightarrow G' \cong C_n$ or D_n .

But $G \cong G'$ so $G \cong C_n$ or D_n some n □

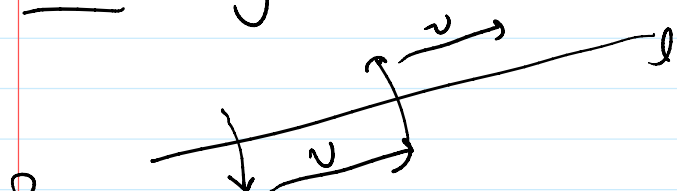
Defⁿ $G \subseteq M_2$ a subgp is said to be discrete if

$\exists \varepsilon > 0$ so that

- if $t_a \in G$ and $t_a \neq 1$ then $\|a\| \geq \varepsilon$

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- if $\underbrace{t_a p_0}_{\text{rotation by } \phi \text{ about some pt in } \mathbb{R}^2} \in G$ with $p_0 \neq 1$ then $|\phi| \geq \varepsilon$.

Note: Why no condition on glide reflections?



Reason: If $t_v r_l$ is a glide reflection,

(r_l is reflection across the line l and v is parallel to l)

$$\text{then } (t_v r_l)^2 = t_v r_l t_v r_l = t_{2v} r_l^2 = t_{2v}$$

So, by the condition above, if $t_v \neq 1$, $\|2v\| \geq \varepsilon$

$$\text{so } \|v\| \geq \frac{\varepsilon}{2}.$$

Will do: Some work towards classifying the discrete subgroups of M_2 .

Recall: $\varphi: M_2 \rightarrow O_2$ gp homom.

$$t_a p \mapsto p,$$

- Suppose $G \subseteq M_2$ is discrete. Set $\bar{G} = \varphi(G) \subseteq O_2$.

$\bar{G}$ is called the "point gp".

- Set $L = \{v \in \mathbb{R}^2 : t_v \in G\} \simeq \ker(\varphi)$

Have: G discrete $\Rightarrow \exists \varepsilon > 0$ so that $v \in L$ and $v \neq 0$ then $\|v\| \geq \varepsilon$.

Have: L discrete $\Rightarrow \exists \varepsilon > 0$ so that $v \in L$ and $v \neq 0$ then $\|v\| \geq \varepsilon$.

Defⁿ Suppose $M \subseteq (\mathbb{R}^n, +)$ is a subg. M is said to be discrete if $\exists \varepsilon > 0$ so that if $m \in M$, $m \neq 0$ then $\|m\| \geq \varepsilon$.

Thm If $G \subseteq M_2$ is discrete, then $L \subseteq (\mathbb{R}^2, +)$ is discrete.

Thm Suppose $L \subseteq (\mathbb{R}^2, +)$ is discrete. Then either

- $L = \{0\}$
- $L = \mathbb{Z}a = \{ma : m \in \mathbb{Z}\}$ for some $a \in \mathbb{R}^2$, $a \neq 0$.
- $L = \mathbb{Z}a + \mathbb{Z}b = \{ma + nb : m, n \in \mathbb{Z}\}$ for some lin. ind. elts $a, b \in \mathbb{R}^2$.

Lemma Suppose $L \subseteq (\mathbb{R}^2, +)$ is discrete.

- A bnded region of $\mathbb{R}^2$ contains finitely many elts of L
- If $L \neq \{0\}$, then $\exists a \in L$, $a \neq 0$ so that $\|a\| \leq \|v\|$ for all $v \in L$, $v \neq 0$.

Such a vector a is said to be of minimal length.

Pf: a) Any sufficiently small sq. contains at most one elt of L because if $v_1, v_2 \in L$, $v_1 \neq v_2$ then $\|v_1 - v_2\| \geq \varepsilon$

a) follows because a bounded region is covered by finitely many such small squares.

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b) This follows from part a). $\square$

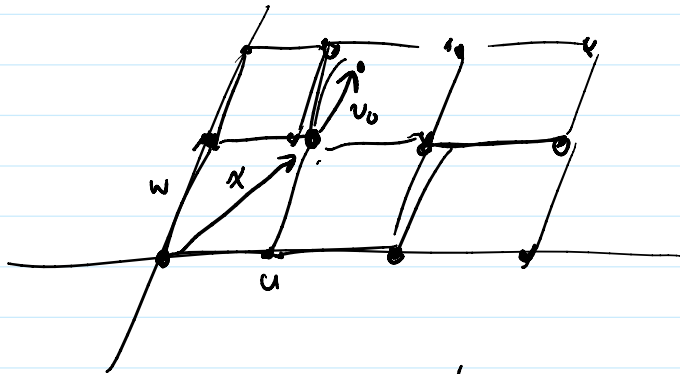
Lemma: Suppose $u, w \in \mathbb{R}^2$ are a basis. Define $L = \mathbb{Z}u + \mathbb{Z}w$

Then any vector $v \in \mathbb{R}^2$ can be written uniquely as

$$v = x + v_0 \quad \text{where } x \in L \text{ and}$$

$$v_0 \in \{ \alpha u + \beta w : 0 \leq \alpha < 1, 0 \leq \beta < 1 \}.$$

Picture:



Pf: Because u, w is a basis,

$$v = ru + sw \quad \text{for } r, s \in \mathbb{R}$$

$$= (m + \alpha)u + (n + \beta)w \quad \text{with } m, n \in \mathbb{Z} \\ 0 \leq \alpha, \beta < 1.$$

$$= \underbrace{mu + nw}_{\in L} + \underbrace{(\alpha u + \beta w)}_{v_0}.$$

$\square$

Pf of Thm $L \subseteq (\mathbb{R}^2, +)$ is discrete.

a) If $L = \{0\}$ we're done

b) Suppose $\exists a' \in L, a' \neq 0$. Either

b) Suppose $\exists a' \in L, a' \neq 0$. Either

1) $L \subseteq \mathbb{R}a'$ or

2) $L \not\subseteq \mathbb{R}a'$.

In case 1, L is isomorphic to a discrete subgroup of $\mathbb{R}$
 $\Rightarrow L = \mathbb{Z}a$ for some $a \in \mathbb{R}a'$.

In case 2, L contains a basis of $\mathbb{R}^2$.

By applying a linear operator S to L , can assume

$$L \cap \mathbb{R}a' = \mathbb{Z} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ we choose so that } S(a) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Consider $T = \left\{ b' = \begin{pmatrix} b'_1 \\ b'_2 \end{pmatrix} \in L : b'_2 > 0 \right\}$.

Claim: T is nonempty. Moreover, $\exists b \in T, b = \begin{pmatrix} b'_1 \\ b'_2 \end{pmatrix}$ so that
 $b_2 \leq b'_2$ for all $b' = \begin{pmatrix} b'_1 \\ b'_2 \end{pmatrix} \in T$

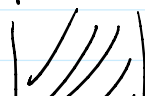


Pf: • T is nonempty b/c L contains a basis of $\mathbb{R}^2$
and $b' \in L \Rightarrow -b' \in L$

• Note that if $b' \in T$ then $b' + m \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} b'_1 + m \\ b'_2 \end{pmatrix} \in T$

• By choosing m appropriately, $0 \leq b'_1 + m \leq 1$

• Now it's clear that $0 \leq b'_1 + m \leq 1$



by choosing m appropriately, $0 \leq m \leq 1$.

Now, it's clear that there are finitely many elts of L in this region



$$\Rightarrow \exists b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \text{ with } 0 \leq b_1 < 1 \text{ and } b_2 \geq 0 \text{ and minimal.}$$

Claim: $L = \mathbb{Z}a + \mathbb{Z}b, \quad a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Pf: Clear that $\mathbb{Z}a + \mathbb{Z}b \subseteq L$. Suppose $c \in L$.

Then by Lemma, $c = (ma + nb) + \alpha a + \beta b$ with $m, n \in \mathbb{Z}, \quad 0 \leq \alpha, \beta < 1$.

$$\text{Then } \underbrace{c - (ma + nb)}_{\in L} = \alpha a + \beta b = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \alpha + \beta b_1 \\ \beta b_2 \end{pmatrix} \in T.$$

But: $\beta b_2 < b_2 \Rightarrow \beta = 0$. if $\beta \neq 0$.

$$\Rightarrow c - (ma + nb) = \alpha a \in \mathbb{R}a \cap L = \mathbb{Z}a$$

$$\Rightarrow \alpha = 0.$$

$$\Rightarrow c = ma + nb \in \mathbb{Z}a + \mathbb{Z}b.$$

□